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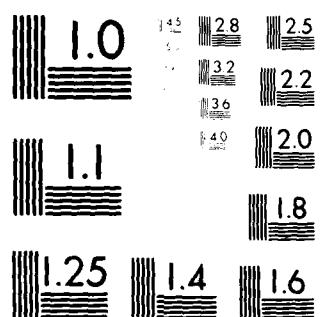
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PARAMETER ESTIMATION TECHNIQUES
for
NONLINEAR DISTRIBUTED PARAMETER SYSTEMS**

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PARAMETER ESTIMATION TECHNIQUES
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NONLINEAR DISTRIBUTED PARAMETER SYSTEMS

H.T. Banks and K. Kunisch

ABSTRACT

Methods for estimating system parameters are discussed for a class of partial differential equations. We develop schemes based on modal subspace approximations in some detail and include numerical examples.

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1. Introduction

In this paper we study approximation methods for linear and nonlinear partial differential equations and associated parameter identification problems. As will be seen, many of the ideas are classical in nature, but the proposed operator-theoretic approach is appealing for its conciseness and generality. The parameter identification problem is explained in Section 2. In Section 3 we develop general nonlinear approximation results which are subsequently used for modal approximations of certain classes of hyperbolic and parabolic partial differential equations. In the fourth section the theory developed so far is employed to approximate the (infinite-dimensional state) parameter identification problem by a sequence of problems for ordinary differential equations. We have tested the modal approximation scheme in numerical experiments for parameter identification in hyperbolic equations; some of our findings are reported in Section 5. Detailed proofs, further numerical results, and a discussion of relevant literature on these problems will appear in a forthcoming paper.

2. The Identification Problem

We consider the abstract Cauchy problem

$$(2.1) \quad \begin{aligned} \dot{u}(t) &= A(q)u(t) + F(q, t, u(t)), \quad t > 0 \\ u(0) &= u_0(q) \end{aligned}$$

where, for each $q \in Q \subset \mathbb{R}^k$, $A(q)$ is the infinitesimal generator of a linear C_0 -semigroup $\{T(t; q)\}_{t \geq 0}$ on a Hilbert space $X(q)$. The inner product and norm in $X(q)$ will be denoted by $\langle \cdot, \cdot \rangle_q$ and $|\cdot|_q$ respectively, although on occasion we shall omit the subscript q . Recall that $t \mapsto u(t)$ is called a mild solution of (2.1) if

$$u(t) = T(t; q)u_0(q) + \int_0^t T(t-\sigma; q)F(q, \sigma, u(\sigma))d\sigma.$$

Conditions on F that guarantee existence of mild solutions to (2.1) will be given below. For the relationship between mild and strong solutions of (2.1) one may consult [4]; here we only note that in many specific instances (i.e., hypotheses on $A(q)$ and/or F and/or u_0) mild solutions are in fact strong solutions.

We assume in our discussions that $X(q)$ is a function space of \mathbb{R}^n -valued "functions" (or the usual Lebesgue equivalence classes) defined on a fixed interval $[0, 1]$ and thus we shall also use the notation $u(t, x; q)$ or $u(t, \cdot; q)$ to denote solutions of (2.1). We shall present approximation techniques for parameter identification problems; these will be discussed in the context of a typical least squares problem. For example, at points $\{x_j\}$, $0 \leq x_1 < x_2 < \dots < x_\ell \leq 1$, and times $\{t_i\}$, $0 \leq t_1 < t_2 \dots < t_r \leq T$, we might be given s-vector

measurements, $s \leq n$, of the n -dimensional "state", which are denoted by $\hat{y}_i \in \mathbb{R}^{ls}$, $i = 1, \dots, r$. These are assumed to represent measurements of $C(q)\xi(t_i; q)$ where $\xi(t_i; q) \equiv \text{col}(u(t_i, x_1; q), \dots, u(t_i, x_l; q))$ and $C(q)$ is an $(sl) \times (nl)$ matrix depending continuously on q . A typical identification problem (ID) is then given by:

(ID): Minimize

$$J(q) = \sum_{i=1}^r |C(q)\xi(t_i; q) - \hat{y}_i|^2$$

over $q \in Q$.

We note that often one cannot observe all components of the state u and hence it is necessary (in order to conform with reality) to use the matrix $C(q)$ in the problem formulation and analysis. We also remark that the point evaluations (at x_j) used to define $\xi(t_i; q)$ above may be meaningful only in specific instances, depending on the equation (2.1), the space X and the initial data u_0 . For the special cases presented below, one can verify that the mild solutions we consider do yield functions for which point evaluation is a valid operation. The least squares criterion that is used in defining (ID) is just one of several possible choices of criteria that might be used in identification or parameter estimation problems and it will be obvious that our discussions and analysis here can be extended to cover other types of functionals J which satisfy hypothesis (H10) of Section 4.

As indicated above we approximate (ID) by a sequence of problems (ID^N) , each of which can be solved with standard numerical procedures.

To formulate the approximating problems, we take for each q a sequence of closed linear subspaces $X^N(q)$ of $X(q)$ endowed with the topology induced by $X(q)$. The orthogonal projections of $X(q)$ onto $X^N(q)$ are denoted by $\pi^N(q)$. We then define the operators $A^N(q) : X(q) \rightarrow X^N(q)$ approximating $A(q)$ by $A^N(q) = \pi^N(q)A(q)\pi^N(q)$ where $\text{Dom}(A^N(q)) = X(q)$. This form of approximating operators, which is a classical one (e.g., see [5, p.369]), has recently proved to be useful in deriving approximation methods for delay-differential equations [1]. We stress that this formulation entails the implicit assumption $X^N(q) \subset \text{Dom}(A(q))$. The projections of F onto the subspaces are given by $F^N(q,t,v) \equiv \pi^N(q)F(q,t,v)$ for each $(q,t,v) \in Q \times [0,T] \times X$ (we shall assume throughout that the spaces $X(q)$ are set-wise all the same set X). The family of approximating equations is therefore given by

$$(2.2) \quad \begin{aligned} \dot{v}(t) &= A^N(q)v(t) + F^N(q,t,v(t)), \quad t > 0 \\ v(0) &= \pi^N(q)u_0(q). \end{aligned}$$

The hypotheses on F given below will insure existence of mild solutions u^N (the notation $u^N(t), u^N(t;q)$ and $u^N(t,x;q)$ will all be used in the sequel) of (2.2). Since $X^N(q)$ is invariant under $A^N(q)$, (2.2) is easily seen to be an initial value problem in the subspace $X^N(q)$. In the event that $X^N(q)$ is finite-dimensional, (2.2) is equivalent to a system of ordinary

differential equations for the generalized Fourier coefficients of the representation for v relative to a chosen basis for $X^N(q)$. In our discussions we make the assumption:

(H1) All elements of $X^N(q)$ are piecewise continuous functions on $[0,1]$.

Defining $\xi^N(t_i; q) \equiv \text{col}(u^N(t_i, x_1; q), \dots, u^N(t_i, x_\ell; q))$, we formulate the approximate identification problems (ID^N) corresponding to (ID) by:

(ID^N) : Minimize

$$J^N(q) = \sum_{i=1}^r |C(q)\xi^N(t_i; q) - \hat{y}_i|^2$$

over $q \in Q$.

Before discussing the existence of solutions to problems such as (ID) and (ID^N) and their relationship, we present convergence results for the approximation of (2.1) by (2.2) in a form readily applicable to the identification problems.

We shall call $\{q, X^N(q), \pi^N(q), A^N(q), F^N(q)\}$ an approximation scheme for (2.1) if $q \in Q \subset R^k$, $X^N(q)$ is a sequence of subspaces of $X(q)$ and $\pi^N(q), A^N(q), F^N(q)$ are maps $\pi^N(q): X(q) \rightarrow X^N(q)$, $A^N(q): X(q) \rightarrow X^N(q)$, $F^N(q): [0, T] \times X(q) \rightarrow X^N(q)$. Such a scheme will be said to be convergent if for any $q^N, \bar{q} \in Q$, $\lim_{N \rightarrow \infty} q^N = \bar{q}$ implies that corresponding mild solutions $u^N(t; q^N)$ of (2.2) converge to a mild solution $u(t; \bar{q})$ of (2.1).

3. Modal Approximations: Convergence Results

In this section we present a convergence theorem for non-linear systems (2.1), (2.2) and then discuss modal approximation schemes for classes of hyperbolic and parabolic equations. We shall refer to a number of hypotheses which we present now.

(H2) For each $q \in Q$, $A(q)$ generates a linear C_0 -semigroup $T(t; q)$.

(H3) The spaces $X(q)$, $q \in Q$, are set-theoretically equal and $A^N(q)$ generates a C_0 -semigroup $T^N(t; q)$ on $X(q)$. Moreover, $q^N \rightarrow \bar{q}$ implies $|T^N(\cdot; q^N)z - T(t; \bar{q})z|_{q^N} \rightarrow 0$ as $N \rightarrow \infty$ for all $z \in X(\bar{q})$, uniformly in t on compact subsets of $[0, \infty)$. There exist constants M and ω independent of N and q such that $|T(t; q)| \leq M e^{\omega t}$ and $|T^N(t; q)| \leq M e^{\omega t}$.

(H4) The set $Q \subset \mathbb{R}^k$ is compact.

(H5) The spaces $X(q)$ are uniformly topologically isomorphic. That is, there exists a real constant K such that $|v|_{\tilde{q}} \leq K |v|_q$ for all q, \tilde{q} in Q .

(H6) The projections $\pi^N(q): X(q) \rightarrow X^N(q)$ satisfy: For any sequence $\{q^N\}$ in Q with $q^N \rightarrow \bar{q}$, one has $|\pi^N(q^N)z - z|_{q^N} \rightarrow 0$ as $N \rightarrow \infty$ for each $z \in X(\bar{q})$.

(H7) The nonlinear function $F: Q \times [0, T] \times X(q) \rightarrow X(q)$ satisfies:

- (i) For each continuous function $u:[0,T] \rightarrow X(q)$,
the map $t \mapsto F(q,t,u(t))$ is measurable for
each $q \in Q$.
- (ii) There exists a function \tilde{k}_1 in $L_2(0,T;R)$ such
that

$$|F(q,t,u_1) - F(q,t,u_2)|_q \leq \tilde{k}_1(t)|u_1 - u_2|_q$$

for all $q \in Q, u_1, u_2 \in X(q)$.

- (iii) There exists a function \tilde{k}_2 in $L_2(0,T;R)$ such
that

$$|F(q,t,0)|_q \leq \tilde{k}_2(t)$$

for all $q \in Q$.

- (iv) For each $(t,u) \in [0,T] \times X(q)$, the map
 $q \mapsto F(q,t,u)$ is continuous.

We remark that to be more precise, we should have written $|\pi^N(q^N)z - \mathcal{J}^N z|_{q^N} \rightarrow 0$, where \mathcal{J}^N is the canonical isomorphism of $X(\bar{q})$ onto $X(q^N)$, in (H6) (a similar adjustment should be made in (H3)). However, we suppress this notation throughout in as much as there will be no confusion in light of the assumed set-wise equality of the $X(q)$, $q \in Q$. We further remark that (H5) implies uniform (in q and N) boundedness of $|\pi^N(q)|_q$.

Theorem 3.1: Suppose that (H2) - (H7) obtain and that $\{q^N\}$ is any sequence in Q with $q^N \rightarrow \bar{q}$. Further assume that

$|u_0(q^N) - u_0(\bar{q})|_{q^N} \rightarrow 0$ as $N \rightarrow \infty$. Then for each $q \in Q$, unique mild solutions $u(t; q)$ and $u^N(t; q)$ of (2.1) and (2.2) exist and we have $|u^N(t; q^N) - u(t; \bar{q})|_{q^N} \rightarrow 0$ as $N \rightarrow \infty$ for each $t \in [0, T]$.

Corollary 3.1: If (H7) is strengthened so that one assumes $(\sigma, v) \mapsto F(\bar{q}, \sigma, v)$ is continuous on $[0, T] \times X(\bar{q})$, then the convergence $u^N(t; q^N) \rightarrow u(t; \bar{q})$ of Theorem 3.1 is uniform in t on $[0, T]$.

Example 3.1. (Hyperbolic equations).

We consider here the equation

$$(3.1) \quad u_{tt} = q_1 u_{xx} + q_2 u_t + q_3 u + f(q_6^1, \dots, q_6^m, t, \cdot, u)$$

with initial and boundary conditions

$$(IC) \quad \begin{aligned} u(0, x) &= \sum_{i=1}^m q_4^i \phi_i(x), \\ u_t(0, x) &= \sum_{i=1}^m q_5^i \psi_i(x), \quad 0 \leq x \leq 1, \end{aligned}$$

$$(BC) \quad u(t, 0) = u(t, 1) = 0, \quad t \geq 0,$$

where $u(t, x)$ and q_i, q_i^j are scalars.

Remark 3.1: Although in (3.1), (IC), (BC) we do not explicitly allow nontrivial boundary conditions (possibly containing parameters), it can easily be seen that such situations are included in our formulation. For consider

$$(3.2) \quad u_{tt} = q_1 u_{xx}$$

with the initial conditions and boundary conditions

$$\begin{aligned} (\widetilde{IC}) \quad u(0, x) &= q_4 \phi(x) \\ u_t(0, x) &= q_5 \psi(x) \end{aligned}$$

$$(\widetilde{BC}) \quad u(t, 0) = q_7 b_1(t), \quad u(t, 1) = q_8 b_2(t),$$

where b_1, b_2 are twice continuously differentiable functions.

Then the usual transformation given by $w(t, x) = u(t, x) - (1-x)q_7 b_1(t) - xq_8 b_2(t)$ transforms (3.2), (\widetilde{IC}), (\widetilde{BC}) into the problem

$$\begin{aligned} w_{tt} &= q_1 w_{xx} - (1-x)q_7 b_{1tt} - xq_8 b_{2tt} \\ w(0, x) &= q_4 \phi(x) - (1-x)q_7 b_1(0) - xq_8 b_2(0) \\ w_t(0, x) &= q_5 \psi(x) - (1-x)q_7 b_{1t}(0) - xq_8 b_{2t}(0) \\ w(t, 0) &= w(t, 1) = 0, \end{aligned}$$

which is a problem that is a special case of the formulation (3.1), (IC), (BC) above.

To treat (3.1), (IC), (BC) we first rewrite (3.1) as an abstract equation in the usual manner employing the operator $\Delta = \frac{\partial^2}{\partial x^2}$

in $H^0 = L_2(0,1; R)$. The Sobolev spaces H^i considered here and below will consist of R^1 -valued functions on $[0,1]$ and we suppress this notation hereafter. With Δ defined on $\text{Dom}(\Delta) = H_0^1 \cap H^2$, we include the boundary conditions (BC) and are also able to establish that Δ is self-adjoint and $\langle -\Delta u, u \rangle \geq |u|^2$ for all $u \in \text{Dom}(\Delta)$. Standard results guarantee existence of $\Delta^{1/2}$ on $\text{Dom}(\Delta^{1/2}) = H_0^1$. We make the following additional assumption:

(HQ) There exist positive numbers q_1^a, q_1^b such that $q \in Q \subset R^k$ implies $q_1^a \leq q_1 \leq q_1^b$.

Then the set $\text{Dom}(\Delta^{1/2})$ endowed with the inner product $\langle u, w \rangle_{q_1} = \langle q_1 u_x, w_x \rangle_{L_2}$ is a Hilbert space which we denote by $V(q_1)$. Finally, $H(q_1) \equiv V(q_1) \times H^0$ with the product topology is a Hilbert space $X(q)$ in which we can now rewrite (3.1), (IC), (BC) as

$$(3.3) \quad \begin{aligned} \frac{d}{dt} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} &= \mathcal{A}(q) \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + F(q, t, u(t)), \quad t > 0 \\ \begin{pmatrix} u(0) \\ v(0) \end{pmatrix} &= \begin{pmatrix} \sum q_4^i \phi_i \\ \sum q_5^i \psi_i \end{pmatrix}, \end{aligned}$$

where $(\phi_i, \psi_i) \in H(q_1)$, $\text{Dom}(\mathcal{A}(q)) = (H_0^1 \cap H^2) \times H_0^1$,

$$\mathcal{A}(q) = \begin{pmatrix} 0 & 1 \\ q_1 \Delta + q_3 & q_2 \end{pmatrix}, \text{ and } F(q, t, u(t)) = \begin{pmatrix} 0 \\ f(q_6, t, \cdot, u(t, \cdot)) \end{pmatrix}.$$

Here $q_j = (q_j^1, \dots, q_j^m)$ for $j = 4, 5, 6$ and $q = (q_1, q_2, q_3, q_4, q_5, q_6)$ is restricted to the fixed set $Q \subset \mathbb{R}^{3m+3}$.

We see immediately that in view of assumption (HQ), the spaces $H(q_1)$ are all topologically isomorphic; indeed for any pair \tilde{q}_1, \hat{q}_1 in $[q_1^a, q_1^b]$ we find $|z|_{H(\tilde{q}_1)} \leq K|z|_{H(\hat{q}_1)}$ with $K = (q_1^b/q_1^a)^{1/2}$ so that (H5) is satisfied. We discuss (H3), (H6) for a specific choice (so-called modal approximations) of finite dimensional subspaces and operator approximations. We refer the reader to [5, pp. 247-250] for the relevant background material needed in our development.

The operator Δ is selfadjoint with compact resolvent and the eigenvectors $\{\tilde{\phi}_j\}_{j=1}^\infty$ and $\{\phi_j\}_{j=1}^\infty$ where $\tilde{\phi}_j(x) = \frac{\sqrt{2}}{j\pi} \sin j\pi x$ and $\phi_j(x) = \sqrt{2} \sin j\pi x$ constitute complete orthonormal sets (CONS) for $V(1) = H_0^1$ and H^0 respectively. We define the modal subspaces $X^N(q) = H^N(q_1)$ of $H(q_1)$ by

$$H^N(q_1) = \text{span} \left\{ \begin{pmatrix} \tilde{\phi}_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} \tilde{\phi}_N \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \phi_N \end{pmatrix} \right\} .$$

We note that $\bigcup_{j=1}^\infty \left\{ \begin{pmatrix} \tilde{\phi}_j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \phi_j \end{pmatrix} \right\}$ forms a CONS for $H(1)$ and a complete orthogonal (but not normal) set for $H(q_1)$, $q_1 \neq 1$.

The modal approximations $\mathcal{A}^N(q)$ for $\mathcal{A}(q)$ are defined by $\mathcal{A}^N(q) = \pi^N(q_1) \mathcal{A}(q) \pi^N(q_1)$, where $\pi^N(q_1)$ is the canonical projection of $H(q_1)$ onto $H^N(q_1)$. The requirement that $H^N(q_1) \subset \text{Dom}(\mathcal{A}(q))$ is trivially seen to be true here.

Theorem 3.2. Let (HQ) obtain and let $q^N, \bar{q} \in Q \subset R^{3m+3}$ be such that $q^N \rightarrow \bar{q}$ as $N \rightarrow \infty$. Then $\mathcal{A}(\bar{q})$ and $\mathcal{A}^N(q^N)$ generate C_0 -semigroups $T(t; \bar{q})$ and $T^N(t; q^N)$ on $H(\bar{q}_1)$ and $H(q_1^N)$ respectively. Furthermore, there is a constant $\omega \in R^1$ independent of N such that $|T(t; \bar{q})| \leq e^{\omega t}$, $|T^N(t; q^N)| \leq e^{\omega t}$ for $t \geq 0$, and for each $z \in H(\bar{q}_1)$, $|T^N(t; q^N)z - T(t; \bar{q})z| \xrightarrow[q^N]{} 0$ as $N \rightarrow \infty$, uniformly in t on compact subsets of $[0, \infty)$.

The proof of this theorem can be given employing a generalized version of the Trotter-Kato theorem (see [3]) and the spectral theorem. One can readily establish dissipativeness of $\mathcal{A}(q) - \omega I$ in $H(q)$ for an appropriately chosen ω . A consequence of (H5) is that one actually obtains $T^N(t; q^N)z \rightarrow T(t; \bar{q})z$ in $H(1)$.

Turning next to the nonlinear equation (3.3), we let $\tilde{Q} = \{q_6 \in R^m | q \in Q\}$ and make the following hypotheses on f .

(H7*) The nonlinear function $f: \tilde{Q} \times [0, T] \times [0, 1] \times R^1 \rightarrow R^1$ satisfies:

- (i) For each $(q_6, u) \in \tilde{Q} \times R^1$, the map $(t, x) \mapsto f(q_6, t, x, u)$ is measurable.
- (ii) There exists k_1 in $L_2(0, T; R)$ such that $|f(q_6, t, x, u_1) - f(q_6, t, x, u_2)| \leq k_1(t)|u_1 - u_2|$ for all $q_6 \in \tilde{Q}$, $t \in [0, T]$, $x \in [0, 1]$, and $u_1, u_2 \in R^1$.
- (iii) There exists k_2 in $L_2([0, T] \times [0, 1]; R)$ such that $|f(q_6, t, x, 0)| \leq k_2(t, x)$ for all $q_6 \in \tilde{Q}$.
- (iv) For each (t, x, u) in $[0, T] \times [0, 1] \times R^1$, the map $q_6 \mapsto f(q_6, t, x, u)$ is continuous.

It is not difficult to see that if one defines F by $F(q, t, z) = \text{col}(0, f(q, t, \cdot, u(\cdot)))$ for $z = \text{col}(u, v)$ in $H(q_1)$, the conditions (H7*) for f imply (H7) for F . Furthermore, it is rather easy to establish that the projections $\pi^N(q_1)$ are in fact independent of q_1 so that completeness of the $\tilde{\phi}_j, \phi_j$ along with (H5) yield (H6). Since (H2) and (H3) follow from Theorem 3.2, we may apply Theorem 3.1 to obtain convergence of the modal approximation scheme $\{q, H^N(q_1), \pi^N(q_1), \alpha^N(q), F^N(q)\}$ for hyperbolic systems (3.1), (IC), (BC).

Example 3.2. (Parabolic equations)

For our second class of examples we consider parabolic equations

$$(3.4) \quad \begin{aligned} u_t &= \frac{q_1}{k}(pu_x)_x + q_2u + f(q_4^1, \dots, q_4^m, t, \cdot, u) \\ u(0, x) &= \sum_{i=1}^m q_3^i \phi_i(x), \quad 0 \leq x \leq 1, \end{aligned}$$

subject to the boundary conditions

$$(3.5) \quad R_j u(t, \cdot) = 0 \quad \text{for } j = 1, 2.$$

Here we assume $\phi_i \in H^0$, $u(t, x) \in R^1$, and $q = (q_1, q_2, q_3, q_4)$ with $q_j = (q_j^1, \dots, q_j^m)$, for $j = 3, 4$. The operators R_j defining the boundary conditions have domain H^2 and are given by

$$R_j v = \alpha_{j1}v(0) + \alpha_{j2}v'(0) + \alpha_{j3}v(1) + \alpha_{j4}v'(1)$$

for $v \in H^2$. We make the following assumptions on k, p and α_{ij} :

(H8) The functions p, p_x and k are in $C(0,1; \mathbb{R})$ with $k(x) > 0$ and $p(x) > 0$ for $0 \leq x \leq 1$.

(H9) The matrix $\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \end{pmatrix}$ has rank 2 and we have
 $p(0)\{\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}\} = p(1)\{\alpha_{13}\alpha_{24} - \alpha_{14}\alpha_{23}\}.$

We again rewrite our problem as an abstract Cauchy problem and to this end we define the generalized Sturm-Liouville operator $\tilde{\mathcal{A}}(q)$ in H^0 by $\text{Dom}(\tilde{\mathcal{A}}(q)) = \{\phi \in H^2 \mid R_j \phi = 0, j = 1, 2\}$ and $\tilde{\mathcal{A}}(q)\phi = k^{-1}(q_1 p \phi_x)_x + q_2 \phi$. Then (3.4), (3.5) can be written as

$$(3.6) \quad \begin{aligned} \dot{u}(t) &= \tilde{\mathcal{A}}(q)u(t) + F(q, t, u(t)), \quad t > 0, \\ u(0) &= \sum_{i=1}^m q_3^i \phi_i, \end{aligned}$$

where $\phi_i \in H^0$ and $F(q, t, u(t)) = f(q_4, t, \cdot, u(t, \cdot))$. We consider this equation and the operator $\tilde{\mathcal{A}}(q)$ in H^0 with inner product $\langle u, v \rangle = \int_0^1 u(x)v(x)k(x)dx$ and note that (unlike the formulation for Example 3.1) our Hilbert space $X(q) = H^0$ is independent of q in this case.

The equality in (H9) implies that $\tilde{\mathcal{A}}(q)$ is selfadjoint and spectral results for $\tilde{\mathcal{A}}(q)$ (e.g. see [2]) yield existence of a CONS (in H^0) of eigenfunctions $\{\psi_j\}_{j=1}^\infty$ of $\tilde{\mathcal{A}}(q)$ where $\tilde{q} \equiv (1, 0, \dots, 0) \in \mathbb{R}^{2m+2}$. As in Example 3.1, we define the approximating modal subspaces of H^0 by $\tilde{H}^N = \text{span} \{\psi_1, \psi_2, \dots, \psi_N\}$ and let

$\pi^N: H^0 \rightarrow \tilde{H}^N$ denote the canonical projections. This determines in the same manner as before the operators $\tilde{\mathcal{A}}^N(q) = \pi^N \mathcal{A}(q) \pi^N$ and $F^N = \pi^N F$. Using the theory of general Sturm-Liouville operators and the Trotter-Kato theorem one can establish convergence of the corresponding semigroups.

Theorem 3.3. Suppose (H0), (H4), (H8) and (H9) hold and let $q^N, \bar{q} \in Q \subset R^{2M+2}$ be such that $q^N \rightarrow \bar{q}$ as $N \rightarrow \infty$. Then $\mathcal{A}(\bar{q})$ and $\tilde{\mathcal{A}}^N(q^N)$ generate C_0 -semigroups $T(t; \bar{q})$ and $T^N(t; q^N)$ on H^0 that satisfy $|T(t; \bar{q})| \leq e^{\omega t}$, $|T^N(t; q^N)| \leq e^{\omega t}$ for some ω which is independent of N . Furthermore $T^N(t; q^N)z \rightarrow T(t; \bar{q})z$ for each $z \in H^0$ with the convergence uniform in t on compact subsets of $[0, \infty)$.

If, in addition to the hypotheses of Theorem 3.3, one assumes that f satisfies (H7*) (with q_6 replaced by q_4), then Theorem 3.1 holds to yield convergence of the approximating solutions of (3.6) and thus the modal approximation scheme $\{q, \tilde{H}^N, \pi^N, \tilde{\mathcal{A}}^N(q), F^N(q)\}$ is a convergent scheme for (3.4), (3.5).

4. Approximation of the Identification Problem

Returning to identification problems such as those discussed in Section 2, we are now in a position to establish convergence of solutions of the approximate problems (\mathcal{P}_N) of minimizing $J^N(q) \equiv \mathcal{J}(u^N(q), q)$ to those of the problem (\mathcal{P}) of minimizing $J(q) \equiv \mathcal{J}(u(q), q)$ where we make the following assumption on the fit-to-data functional \mathcal{J} .

(H10) The mapping $\mathcal{J}: X(q) \times Q \rightarrow R$ is a continuous functional; here $X(q)$ is endowed with any of the equivalent topologies hypothesized in the standing assumption (H5).

Our results are stated precisely in the form of a theorem.

Theorem 4.1. We assume the hypotheses (H1)-(H7) and (H10) hold and that $q \rightarrow u_0(q)$, $q \rightarrow \pi^N(q)z$ and $q \rightarrow T^N(t;q)z$ are continuous for each $z \in X(q)$ and $t \in [0,T]$. Then

(a) for each N there exists a solution q^N of (\mathcal{P}_N) ,

(b) there exists a subsequence $\{q_j^N\}$ of $\{q^N\}$ converging to some $\bar{q} \in Q$ which is a solution of (\mathcal{P}) .

Moreover, $|u_j^N(t; q_j^N) - u(t; \bar{q})|_{q_j^N} \rightarrow 0$ as $N_j \rightarrow 0$

where u^N, u are solutions of (2.2), (2.1) respectively.

Proof: It is not difficult to argue (using (H3)-(H7) and (H10)) that $q \rightarrow J^N(q)$ is continuous on the compact set Q . Existence of a subsequence $\{q_j^N\}$ with $q_j^N \rightarrow \bar{q}$ follows from (H4). We observe that for any $q \in Q$, one has $J_j^N(q_j^N) \leq J_j^N(q)$. From Theorem 3.1, (H5) and (H10), we see that $J_j^N(q) \rightarrow J(q)$ for each $q \in Q$ and $J_j^N(q_j^N) \rightarrow J(\bar{q})$ so that $J(\bar{q}) \leq J(q)$ for any $q \in Q$. That is, \bar{q} is a solution of (\mathcal{P}) .

If we further assume (HQ), it is quite simple to see that Theorem 4.1 is applicable to identification problems for Examples 3.1 and 3.2. We have conducted numerical investigations for the modal approximation scheme for identification problems with the hyperbolic systems of Example 3.1 and report briefly on some of them in the next section. For these calculations we chose $C(q) = (1,0)$ in the functionals J and J^N of (ID) and (ID^N) of Section 2, thereby enabling one to verify (H10).

5. Numerical Results

In this section we present two examples, deferring a thorough discussion of our numerical experience with modal schemes to the forthcoming paper. A standard IMSL-package employing the Levenberg-Marquardt algorithm was used to solve the parameter identification problems for the approximating ordinary differential equations. "Exact" solutions for the distributed systems below were generated independently by a simple Crank-Nicolson algorithm and some of these values were used for the "data" \hat{y}_i in (ID^N) . We would like to express our deep appreciation to James Crowley for his efforts in developing the software packages employed in our computational experiments.

Example 1. We consider equation (3.1) with $f \equiv 0$:

$$u_{tt} = q_1 u_{xx} + q_2 u_t + q_3 u,$$

where $u(0,x) = q_4 x(1-x)$, $u_t(0,x) = 2q_5 x$ for $0 \leq x \leq \frac{1}{2}$ and $u_t(0,x) = q_5(2-2x)$, for $\frac{1}{2} \leq x \leq 1$, and $u(t,0) = u(t,1) = 0$ for all t . The true model parameters were taken to be $\bar{q} = (1.414, 0, 0, 4, 5)$.

We performed a five dimensional search starting at $q^{N,0} = (1,0,0,1,0)$ for each value of N ; the results obtained by applying the Levenberg-Marquardt procedure for several values of N are given in Table 1.

Table 1.

| N | q_1^N | q_2^N | q_3^N | q_4^N | q_5^N |
|-------------|---------|---------|---------|---------|---------|
| 4 | 1.3863 | 0.0026 | -0.2762 | 3.9876 | 5.0273 |
| 8 | 1.4112 | -0.0035 | -0.0308 | 4.0039 | 5.0195 |
| 16 | 1.4139 | -0.0001 | -0.0025 | 3.9999 | 5.0030 |
| 32 | 1.4139 | 0.0000 | -0.0006 | 4.0001 | 5.0001 |
| True Values | 1.4140 | 0.0 | 0.0 | 4.0 | 5. |

Example 5.2. In this example we present computations for the nonlinear equation

$$u_{tt} = q_1 u_{xx} + q_2 u_t + q_3 u + \frac{u}{1+u},$$

with $u(0,x) = q_4 x(1-x)$, $u_t(0,x) = q_5$ and homogeneous boundary conditions. Table 2 depicts the results corresponding to the "true" model parameters $\bar{q} = (1.414, 0, 1, 2, 0)$. Holding q_2 and q_5 fixed, a three dimensional search for q_1, q_3 and q_4 was performed starting with $q^{N,0} = (1, 0, 0, 1, 0)$.

Table 2.

| N | $\frac{N}{q_1}$ | $\frac{N}{q_3}$ | $\frac{N}{q_4}$ |
|-------------|-----------------|-----------------|-----------------|
| 4 | 1.3876 | 0.7441 | 2.0007 |
| 8 | 1.4091 | 0.9509 | 2.0003 |
| 16 | 1.4124 | 0.9774 | 2.0003 |
| 32 | 1.4137 | 0.9904 | 2.0003 |
| True Values | 1.4140 | 1.0 | 2.0 |

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